Free fall and harmonic oscillations - analysing trampoline jumps

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Abstract. Trampolines can be found in many gardens and also in some playgrounds. They offer an easily accessible vertical motion, including free fall. In this work, the motion on a trampoline is modelled by assuming a linear relation between force and deflection, giving harmonic oscillations for small amplitudes. An expression for the cycle-time is obtained in terms of maximum normalised force from the trampoline and the harmonic frequency. A simple expression is obtained for the ratio between air-time and harmonic period, and the maximum g-factor. The results are compared to experimental results, including accelerometer data showing $7g$ during bounces on a small trampoline in an amusement park play area. Similar results are obtained in a larger garden trampoline and even stronger forces have been measured for gymnastic trampolines.

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1. Introduction

Where do you experience the largest forces in an amusement park? Amusement park standards allow rides that make the rider feel up to 6 times heavier than normal, i.e. $6g$, although many modern roller coasters stay well below 5g. As the Liseberg amusement park in Gothenburg [1] recently asked it's Facebook followers to guess which rides involves the largest forces, it was hardly surprising that the newer roller coasters and a free fall ride were the favourites. However, larger forces - up to $7g$ - were in fact encountered in the small trampoline found in the children's playground area at Liseberg (the "Rabbit Land"). Similar data have also been obtained in a round domestic trampoline, such as the one shown in figure 1, where a spiral toy is used to provide a visual illustration of the forces on the user. (The same toy has been used e.g. to illustrate

Figure 1. A spiral toy used as accelerometer during a trampoline jump. Note how the spiral in the toy is contracted while the jumper is in the air, and expanded at the bottom of the jump. Also note how far down the trampoline is displaced in the last photo.

forces in a swing [2]). When the feet are not in contact with the trampoline, the user experiences free fall ($0g$), and the spiral in the toy is contracted. At the bottom of a the jump, when the trampoline bed experiences the largest displacement, a large force is exerted on the user, as illustrated by the extension of the spiral in the toy. Below, we first analyse the motion during bounces in a trampoline, and then compare the results to data from the trampoline in Liseberg's Rabbit land.

2. From harmonic oscillator to free fall

For a simplified analysis, we neglect energy losses, both in contact with the trampoline and in the air. We consider a person with mass m jumping on a trampoline. As long as the person is in contact with the trampoline, it exerts an upwards force, given by $F(z) = -kz$, where k is the "spring constant" of the trampoline, and z is the vertical displacement from the rest position of the trampoline (positive is upwards). When the person stands still, the force from the trampoline exactly counteracts gravity, i.e. $kz = -mg$. For small oscillations around this equilibrium position, starting at the lowest point at time $t = 0$ the motion can be described by $z(t) = -mg/k - A\cos\omega t$, where $\omega^2 = k/m$. We can thus write

$$z(t) = -g/\omega^2 - A\cos\omega t$$

Velocity and acceleration are obtained by taking derivatives, giving

$$v(t) = A\omega\sin\omega t$$

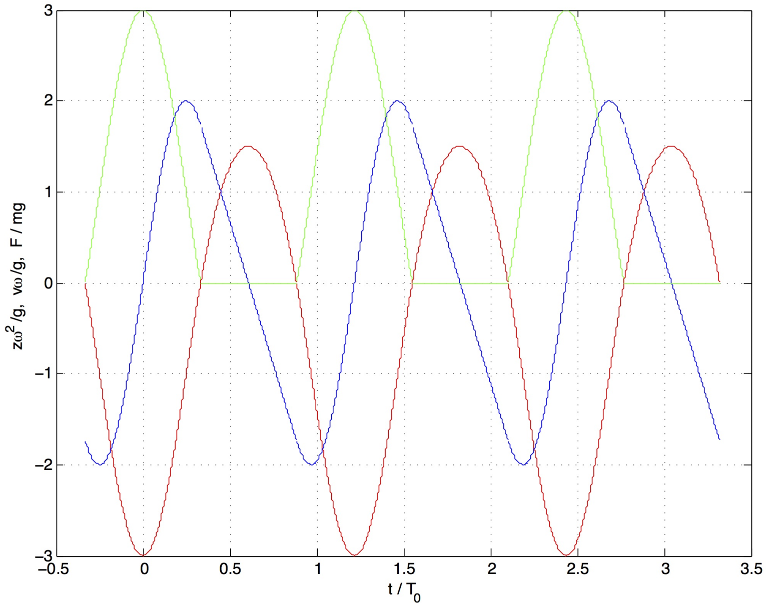

Figure 2. Theoretical values for the elevation, speed and normalised force during a jump reaching $3g$. The green line represents acceleration, the blue line velocity, and the red line displacement. The data are presented using dimensionless quantities.

$$a(t) = A\omega^2 \cos \omega t$$

The period for small oscillations is given by $T_0 = 2\pi/\omega$. As long as Ag/ω^2 the user remains in contact with the trampoline bed and the user's motion is sinusoidal on the z-axis. For larger amplitudes, the separation of the user from the trampoline bed commences he force increases beyond $2g$. As the force increases the user experiences longer and longer periods of $0g$ (air-time).

Like children on a swing, the trampoline user is continually converting energy to reach incrementally higher with each oscillation. While the feet are in contact with the trampoline, the users can add energy to the system, by raising the centre of mass when the forces are largest, e.g. by stretching the legs or swinging the arms. While the feet are in contact with the trampoline, small changes in the body position allows the user to lift the centre of mass and thus increase the maximum potential energy on successive jumps. The details of the movements will not be considered here, but examples can be found e.g. in movie recordings from the 2012 Olympic games [3]. The potential energy is then converted to kinetic energy as the user falls and this kinetic energy in converted into to spring energy as the feet make contact with the trampoline and the user stretches the trampoline bed in a downward direction. As the user falls from greater and greater heights the downward displacement of the trampoline bed becomes larger, causing larger forces from the trampoline on the user. During the contact time, the force from the trampoline makes the speed rapidly decrease to zero at the bottom dead centre, and then increasing again until the user passes the equilibrium position.

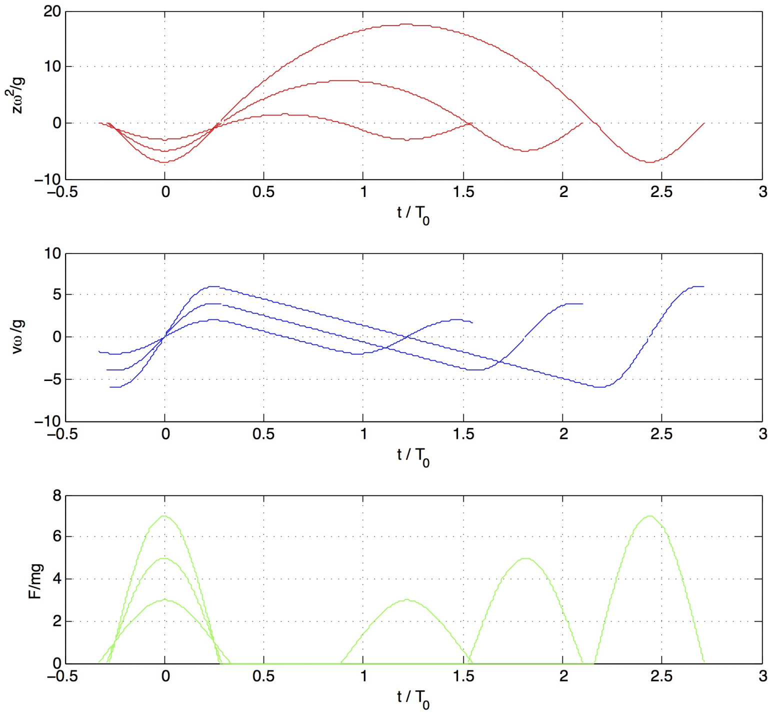

Figure 3. Theoretical values for the displacement, velocity and normalised force during a jump reaching $3g$, $5g$ and $7g$. The green line represents acceleration, the blue line velocity, and the red line displacement.

Since the whole body accelerates, every part of the body experiences strong forces that are analogous to the forces of a stronger gravitational field. These forces are commonly referred to as g-forces. The trampoline doesn't just provide coordination and exercise for the users heart, leg muscles, core strength, long-bones and drain the poisons from the lymphatic system, the elevated gravitational force is applied to each and every cell of the user's body. Each cell is cycled from $0g$ to more than $5g$ each time the trampoline user cycles from weightless to a gravitational field 5 times that of earth. This cycling 'exercises' each and every cell of the user's body.

If the trampoline user experiences Ng, the maximum upward force from the trampoline can be written as Nmg. This corresponds to a maximum downward deflection $z_N = -Nmg/k$, and an amplitude $A = (N-1)mg/k = (N-1)g\omega^2$.

Let us now consider in more detail a jump, starting at the lowest point. The motion during the first part of the jump when the jumper is in contact with the trampoline can still be written as,

$$z(t) = -g/\omega^2 - A\cos\omega t = -\frac{g(1+(N-1)\cos\omega t)}{\omega^2}$$

However, since the trampoline does not pull the jumper downwards, this expression holds only as long as $z(t) < 0$, i. e. for $\omega t < \arccos(-1/(N-1))$. After this time, the displacement becomes positive and only gravity acts on the trampoline user, who will be in free fall until touching the trampoline again. The user then leaves the trampoline

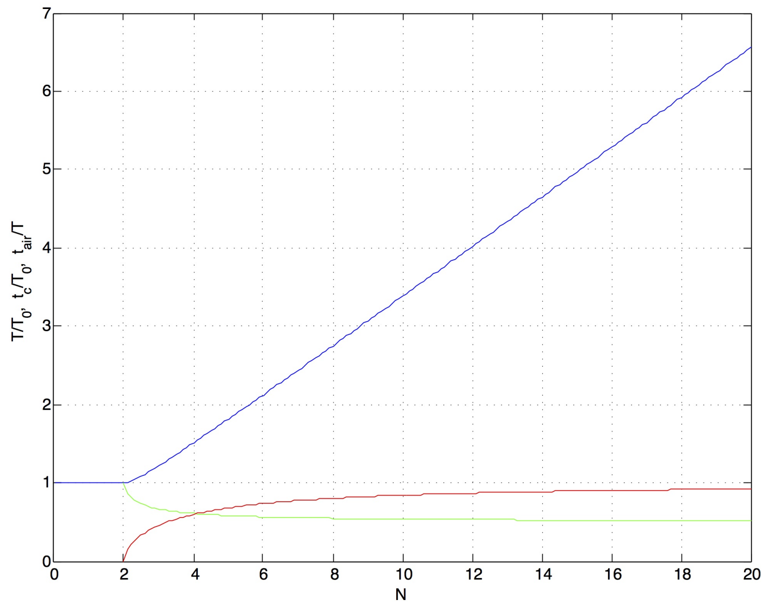

Figure 4. Dependence of the bouncing period for different values of the maximum force, given as a ratio to the harmonic oscillator period, T_0, for amplitudes smaller than mg/k (blue). The green line gives the time in contact with the trampoline. Also shown (red) is the ratio between air-time and cycle-time.

with a speed

$$v_t = A\omega \sin(\arccos \frac{-1}{N-1}) = (N-1)\frac{mg}{k}\sqrt{k/m}\sqrt{1 - \frac{1}{(N-1)^2}}$$

which can be rewritten as

$$v_t = g\sqrt{N^2 - 2N}/\omega$$

This velocity also gives the maximum height during the jump: $h = v^2/2g$. Lighter jumpers have a higher $\omega = \sqrt{k/m}$ and will reach a lower velocity for the same acceleration at the bottom of the jump. Similarly, a stiffer trampoline will give lesser speed for the same maximum acceleration. Neglegting energy losses, the jumper will land again after a time

$$t_{air} = 2\sqrt{N^2 - 2N}/\omega = T_0\sqrt{N^2 - 2N}/\pi$$

Figure 2 shows the displacement, speed and forces during different parts of the bounce for a maximum force of 3g. Figure 3 shows the elevation, speed and force during the motion for a maximum force of $3mg$, $5mg$ and $7mg$. The time in contact with the trampoline during a full cycle is given by

$$t_c = 2\arccos(-1/(N-1))/\omega = T_0 \arccos(-1/(N-1))/\pi$$

For high bounces (large N values), the contact time during a cycle approaches $T_0/2$.

Using the analysis above we find en expression for the total time for one period on the trampoline:

$$T = t_c + t_{air} = T_0 \frac{\arccos(-1/(N-1)) + \sqrt{N^2 - 2N}}{\pi}$$

Figure 4 shows how the ratio T/T_0 for the bounces depends on the maximum force. The ratio of airtime to cycle-time $q = t_{air}/T$ grows with for high bounces and is also shown in figure 4.

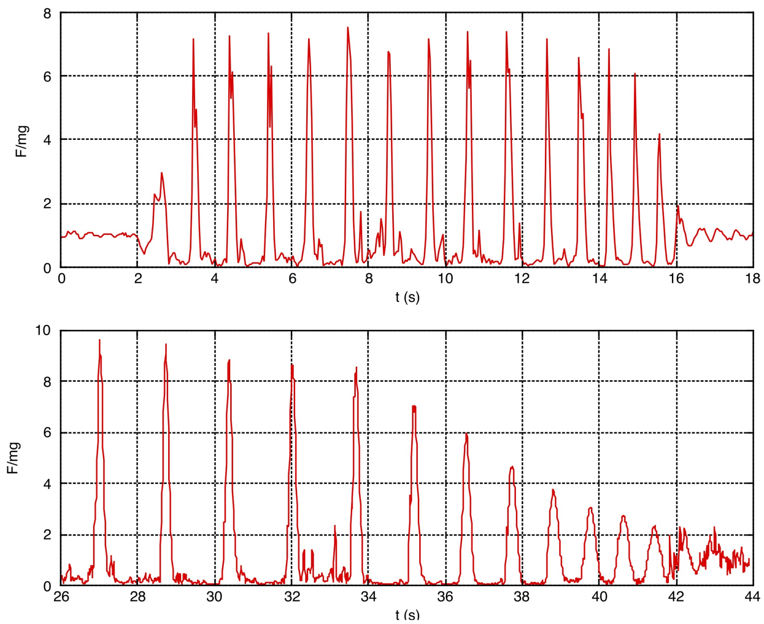

Figure 5. Accelerometer data for bounces on the 1.3m circular trampoline at Liseberg (sampling frequency 25 Hz). In addition, data for a gymnastic trampoline (sampling frequency 100 Hz) has been included, exceeding $9g$, which clearly exhibit the damping and how cycle-time increases with the g-factor. The longer period for around $7g$ in the second graph also shows that the ratio k/m was smaller for that case.

3. Experimental results

As preparation for Liseberg's question about the largest g-force, it was decided to perform a measurement also on the small trampoline (1.3 m diameter) in the children's play area. The data were collected using the Wireless Dynamic Sensor from Vernier [5] placed in a data vest. Using the accompanying program Logger Pro [6], the data could also be synchronised with a movie recording of the jump, as shown in the video abstract accompanying this paper. Figure 5 shows the force acting on a trampoline user during bouncing, both in the small trampoline at Liseberg and in a larger gymnastics trampoline.

It is worth noting that, in spite of the name, an accelerometer does not measure acceleration, but instead the vector $\mathbf{f} = (\mathbf{a} - \mathbf{g})/|g|$. Thus, for free fall, when $\mathbf{a} = \mathbf{g}$,

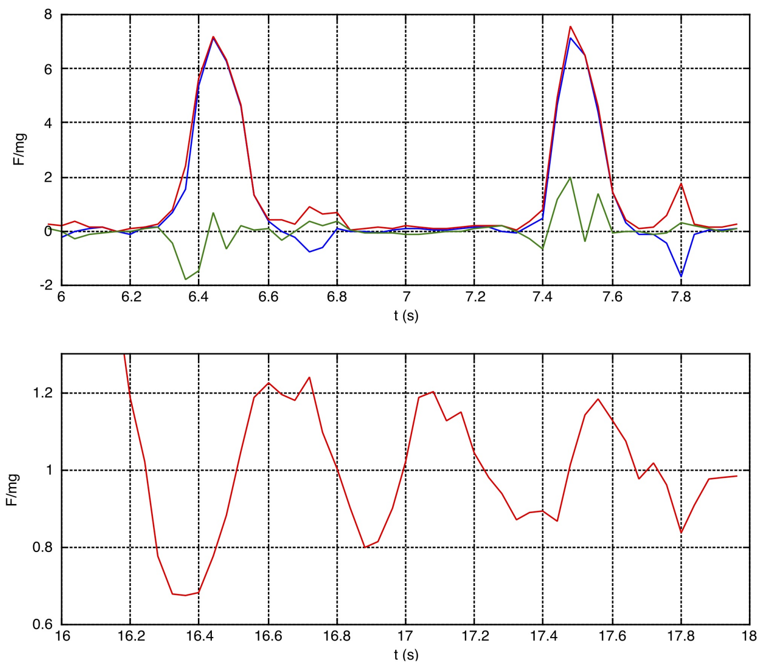

Figure 6. Details of the accelerometer data for the trampoline at Liseberg shown in figure 5. The red curve shows the magnitude of the vector sum of the components, whereas the blue and green curves show the z and x components. The negative x in the beginning of the contact time indicates the user leaned somewhat forward, whereas the negative z values after the contact time may have been obtained by pulling the knees up. Data were collected at 0.04 s intervals (25 Hz).

it becomes zero. As expected, the accelerometer data are essentially zero g when the jumper is in the air, (apart from user movement while air-bourne during the jump). The strongest force acts at the lowest point, where the trampoline mat is most extended. The surprising result was that the largest g-force in the amusement park are to be found in the children's area: around $7g$. To compare the results with the theoretical analysis above, the resonance frequency for small oscillations is needed. Thus, a few small bounces were performed, where the feet remained in contact with the trampoline. Figure 6 shows in more detail a few of the high bounces, as well as the final small bounces. The ratio between these two periods should be directly related to the g-force, as seen in figure 4.

4. Comparison between theory and experiment

From figure 6 we can extract the cycle-times for small oscillations and for the highest bounces. We find that the ratio between these two cycle-times corresponds to a maximum force of about $6.2mg$. In figure 7 the theoretical graph is superimposed on the experimental data. The accelerometer data, however, shows more than $7g$, which can possibly be accounted for by the bouncer giving an extra push at the bottom. Vernier does not specify the quality of the data from the WDSS sensor beyond $6g$ [5], but the relatively smooth data shown in figure 7 does not indicate any obvious problem.

The relation between bounce period and force shown in figure 4 can be used to analyse e.g. the elephant bouncing in [7], where the video-analysis gave a period $T \approx 2.2\,\mathrm{s}$, with a contact time less than $0.3\,\mathrm{s}$. For large accelerations, the contact time is close to $T_0/2$, giving $T_0 \approx 0.6\,\mathrm{s}$ (although the period seems close to $1\,\mathrm{s}$ for the few elephant bounces shown in the movie before the elephant leaves the trampoline.) This ratio would correspond to a maximum force of over $11mg$. Rhett Alain [7] concluded for other reasons that the movie was a fake.

The relation shown in figure 4 can also be applied e.g. to the investigations of different trampoline spring systems by Eager *et al* [4]. In that work, it was found that the average ratio of air-time to cycle-time varied between 0.58 and 0.61 for the three trampolines tested. The largest ratio corresponded to approximately $5g$, slightly larger that expected from the graph.

The analysis leading to the graph in figure 4 is based on the assumption that the deflection on the trampoline is proportional to the force. However, for deflections that are large compared to the trampoline dimensions, the linear approximation is less valid [8]. This may account for the small differences found in the ratio obtained from Eager *et al* [4] and in the experimental data shown in figure 7, which were collected for jumps on the 1.3 m diameter trampoline in the amusement park.

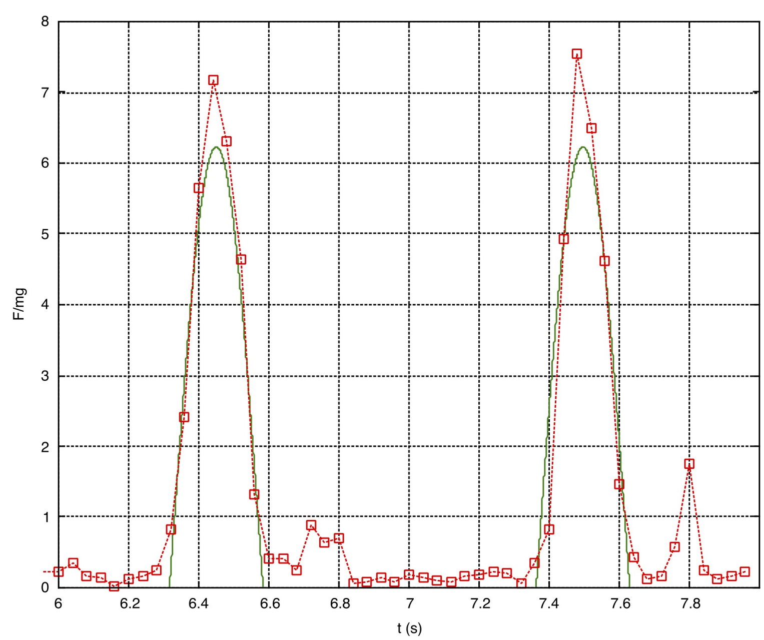

Figure 7. Comparison between theoretical results and accelerometer data. Sampling rate 25 Hz.

5. Discussion

Motion on a trampoline combines two common text-book examples: The harmonic oscillator and free fall. Whereas the free fall is independent of mass, the harmonic oscillator frequency is not. The analysis of the motion opens possibilities for a discussion of what variables influences the motion, and what variables are best suited for presenting the results. Different trampoline stiffnesses and user masses can be combined into a single variable, $\omega = \sqrt{(k/m)}$ or $T_0 = 2\pi/\omega$. The time dependence of displacement, speed and acceleration can then be expressed in terms of ω or T_0, combined with the factor N characterising the maximum force from the trampoline. The maximum g-force on the user can be obtained from the ratio of airtime during the bounce. Trampolines are commonly available and more use should be made of them in physics and science teaching. Studying motion upon the small trampoline at Liseberg will be included as one of the possible activities during future physics days. The trampoline motion is suitable for modelling, as well as successive refinement of the model, using e.g. video analysis and accelerometer data.

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